Design tools for structures adapting to impact loads

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The methodology (based on the so-called Dynamic Virtual Distortion Method) of design of structures exposed to impact loading is presented. Maximization of energy dissipation can be chosen as the objective function for optimal structural adaptation to impact load. The cross-sections of structural members as well as stress levels triggering plastic-like behaviour of energy dissipaters are design parameters. A general formulation of this problem as well as particular cases are discussed.

Key words: *adaptive structures, optimal control, dynamic sensitivity analysis.*

1. Introduction

Adaptive structures (structures equipped with sensor system, as well as controllable semi-active dissipaters, so-called *structural fuses*) with highest ability of adaptation to extreme overloading are discussed.

The dynamic formulation of this problem allows for development of effective numerical tools of optimal design for the best structural crash-worthiness (see [2]). Structures with the highest impact absorption properties can be designed in this way. The proposed optimal design method combines sensitivity analysis with remodelling process and plastic-like stress-limit adaptation, proposing an approach (with material distribution as well as stress limits control) how to design an optimally redesigned structure. The so-called Virtual Distortion Method (see [1]), leading to analytical formulas for gradient calculations, has been used in numerically efficient algorithm.

2. VDM-based dynamic analysis of adaptive structure

In this section we will formulate the VDM-based description of dynamic response of elasto-plastic truss structure. Applying assumed time discretisation, the evolution of strains and stresses (with respect to initial cross-sections A') can be expressed as follows:

$$\varepsilon_i(t) = \varepsilon_i^{\mathrm{L}}(t) + \sum_{\tau \leqslant t} \sum_j D_{ij}^{\mathrm{D}}(t-\tau)\,\Delta\varepsilon_j^0(\tau) + \sum_{\tau \leqslant t} \sum_k D_{ik}^{\mathrm{H}}(t-\tau)\,\Delta\beta_k^0(\tau), \quad (2.1)$$

$$\sigma_i'(t) = E_i\left(\varepsilon_i(t) - \varepsilon_i^0(t) - \beta_i^0(t)\right),$$

$$\sigma_i'(t) = E_i\Bigg[\varepsilon_i^{\mathrm{L}}(t) + \sum_{\tau \leqslant t} \sum_j \left(D_{ij}^{\mathrm{D}}(t-\tau) - \delta_{ij}\right)\Delta\varepsilon_j^0(\tau) \\ + \sum_{\tau \leqslant t} \sum_k \left(D_{ik}^{\mathrm{H}}(t-\tau) - \delta_{ik}\right)\Delta\beta_k^0(\tau)\Bigg], \quad (2.2)$$

where the so-called *dynamic influence matrices* $D_{ij}(t-\tau)$ describe the strain evolution caused in the truss element member i in the time instance t, due to unit virtual distortion impulse generated in member j in the time instant τ. However, in matrix D_{ij}^{D} unit impulse is applied by the Dirac-like function and in matrix D_{ik}^{H} by the Heavyside-like function. The vector denotes the strain evolution due to external loads applied to elastic structure with initial material distribution (unmodified cross-sections of members), $\varepsilon_i^0(t)$ denotes virtual distortions responsible for modification of design variables and $\beta_i^0(t)$ describes plastic-like distortions. Note that the matrix $\mathbf{D}$ stores information about the entire structure properties (including boundary conditions) and describes dynamic (not static) structural response to locally generated impulse of virtual distortion. From now on, we assume that small Latin indices j run through all modified members, and small Latin indices k run through all plastified elements.

Taking advantage of two expressions for the internal forces applied to the so-called *distorted* (2.3) (with modification of material distribution modelled through virtual distortions) and *modified* (2.4) (with redesigned cross-sections form A to A', without imposing virtual distortions) structure:

$$P_i(t) = E_i A_i \left(\varepsilon_i(t) - \varepsilon_i^0(t) - \beta_i^0(t)\right), \quad (2.3)$$

$$P_i(t) = E_i A_i' \left(\varepsilon_i(t) - \beta_i^0(t)\right), \quad (2.4)$$

a formula combining components $\varepsilon_i^0(t)$ and $\beta_i^0(t)$ can be derived, where these components are non-zero only for distorted and/or plastified elements.

If we assume that forces and strains in both structures: *distorted* (2.3) and *modified* (2.4) are the same, the modifications simulated with virtual distortions can be combined with these distortions through the following formula:

$$\varepsilon_i^0(t) = (1 - \mu_i)\left(\varepsilon_i(t) - \beta_i^0(t)\right), \tag{2.5}$$

where $\varepsilon_i(t)$ describes strain in member i in time t, while $\mu_i = \frac{A_i}{A_i'}$ denotes ratio of new cross-section to initial one. Parameter $\mu_i \in \langle 0, 1 \rangle$ specifies size of modification of cross-sections in element i. Condition $\mu_i = 1$ means that there is no change of cross-section in element i, whereas condition $\mu_i = 0$ means that element i has been eliminated.

The formula (2.5) can be rewritten in the following form:

$$\mu_i = \frac{A_i}{A_i'} = \frac{\varepsilon_i(t) - \varepsilon_i^0(t) - \beta_i^0(t)}{\varepsilon_i(t) - \beta_i^0(t)}. \tag{2.6}$$

Now let us substitute the time-dependent strain (2.1) to formula (2.5), getting the following set of equations:

$$\sum_{\tau \leqslant t} \sum_j \left[(1 - \mu_i)D_{ij}^{\mathrm{D}}(t - \tau) - \delta_{ij}\right] \Delta\varepsilon_j^0(\tau)$$

$$+ \sum_{\tau \leqslant t} \sum_k (1 - \mu_i)\left[D_{ik}^{\mathrm{H}}(t - \tau) - \delta_{ik}\right]\Delta\beta_k^0(\tau) = -(1 - \mu_i)\varepsilon_i^{\mathrm{L}}(t). \tag{2.7}$$

To obtain $\Delta\varepsilon_j^0(t)$ for entire time period $t \in \langle 0, T \rangle$, we have to solve (step by step) the set of equations (2.7) for all time instances. However, if we assume that

$$D_{ij}(0) = 0, \tag{2.8}$$

we will get a simple diagonal form given by

$$\Delta\varepsilon_i^0(t) = (1 - \mu_i)\varepsilon_i^{\mathrm{L}}(t) + \sum_{\tau \leqslant t} \sum_k (1 - \mu_i)\left[D_{ik}^{\mathrm{H}}(t - \tau) - \delta_{ik}\right]\Delta\beta_k^0(\tau)$$

$$+ \sum_{\tau < t} \sum_j \left[(1 - \mu_i)\,D_{ij}^{\mathrm{D}}(t - \tau) - \delta_{ij}\right]\Delta\varepsilon_j^0(\tau), \tag{2.9}$$

which requires only computation of the right-hand side expression. If we know the parameter μ_i, the step-by-step determination of the increment $\Delta\varepsilon_i^0(t)$ can be performed making use of the formula (2.9).

In order to take into account elasto-plastic structural behaviour, let us apply bilinear constitutive model with hardening (Fig. 1), given by

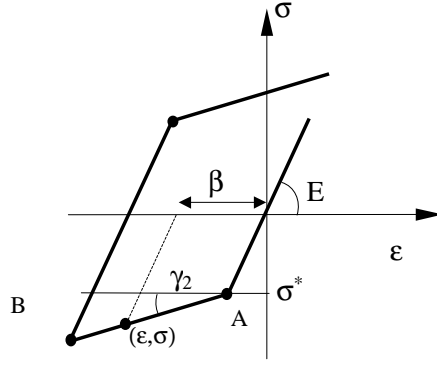$$\sigma_i(t) - \sigma_i^* = \gamma_i\,E_i\left(\varepsilon_i(t) - \varepsilon_i^*\right), \tag{2.10}$$

FIGURE 1. Piece-wise linear constitutive relation for the adaptive structural member.

where σ_i^* denotes plastic yield stress, γ_i denotes hardening parameter and E_i denotes Young's modulus.

For the *modified structure,* the stress formula must be rescaled by parameter μ_i, and the corrected stress equation is given by following relationship:

$$\sigma_i(t) = \frac{\sigma_i'(t)}{\mu_i} = E_i \left(\varepsilon_i(t) - \beta_i^0(t) \right). \tag{2.11}$$

Now, let us substitute strain (2.1) and stress (2.2) to formula (2.10), which leads to the following set of equations:

$$\sum_{\tau \leqslant t} \sum_k \left[(1 - \gamma_i) D_{ik}^{\mathrm{H}}(t - \tau) - \delta_{ik} \right] \Delta \beta_k^0(\tau)$$
$$+ \sum_{\tau \leqslant t} \sum_j (1 - \gamma_i) D_{ij}^{\mathrm{D}}(t - \tau) \Delta \varepsilon_j^0(\tau) = -(1 - \gamma_i) \left(\varepsilon_i^{\mathrm{L}}(t) - \varepsilon_i^* \right). \tag{2.12}$$

Once again let us use assumption (2.8). Then we get the simple form to calculate plastic-like distortion in each time step, which requires only computation of the right-hand side expression given by the formula:

$$\Delta \beta_i^0(t) = (1 - \gamma_i) \left(\varepsilon_i^{\mathrm{L}}(t) - \varepsilon_i^* \right) + \sum_{\tau \leqslant t} \sum_j (1 - \gamma_i) D_{ij}(t - \tau) \Delta \varepsilon_j^0(\tau)$$
$$+ \sum_{\tau < t} \sum_k \left[(1 - \gamma_i) H_{ik}(t - \tau) - \delta_{ik} \right] \Delta \beta_k^0(\tau). \tag{2.13}$$

Formulas (2.8), (2.9) and (2.13) allow us to compute the time evolution of virtual distortions modelling both: assumed remodelling of material distribution as well as adapted plastic-like stress limits.

If there is no plasticity in our problem, then plastic-like distortions are equal to zero and the equation (2.9) takes the following form:

$$\Delta\varepsilon_i^0(t) = (1 - \mu_i)\,\varepsilon_i^{\mathrm{L}}(t) + \sum_{\tau<t}\sum_j \left((1 - \mu_i)D_{ij}^{\mathrm{D}}(t - \tau) - \delta_{ij}\right)\Delta\varepsilon_j^0(\tau). \quad (2.14)$$

Analogously, if there is no remodelling, distortions are equal to zero (the parameter μ_i is equal to one) and equation (2.13), determining plastic-like distortions development, takes the following form:

$$\Delta\beta_i^0(t) = (1 - \gamma_i)\left(\varepsilon_i^{\mathrm{L}}(t) - \varepsilon_i^*\right) + \sum_{\tau<t}\sum_k [(1 - \gamma_i)H_{ik}(t - \tau) - \delta_{ik}]\,\Delta\beta_k^0(\tau).$$
$$(2.15)$$

3. Gradient-based approach using VDM

Let us assume that the objective function is defined as maximization of dissipated energy during the adaptation process, given by the following formula:

$$U_{\max} = \sum_t \sum_i \sigma_i(t)\Delta\beta_i(t)\mu_i A_i' l_i, \quad (3.1)$$

subject to constrains:

$$-\tilde{\beta}_i \leqslant \beta_i \leqslant \tilde{\beta}_i, \quad 0 \leqslant \mu_i \leqslant 1, \quad \mu_i A_i' l_i = \mathrm{const.},$$

where $\tilde{\beta}_i$ denotes the lower and upper limit imposed on plastic-like distortions, $\mu_i A_i' l_i$ denotes the total volume of material, which should be constant during remodelling process.

To make the further analysis more communicative, let us distinguish two particular cases. The first one, dealing with the best material redistribution in all structural members, leads to determination of the following design variables: $\mu_i = \frac{A_i}{A_i'}$. In this case $\Delta\beta_k^0(t) = 0$, and is replaced by $\varepsilon_i(t)$ in the objective function (3.1).

The gradient of this objective function can be calculated analytically and takes the following form:

$$\frac{\partial U}{\partial \mu_m} = \left(\frac{\partial U}{\partial \sigma_p(t)}\frac{\partial \sigma_p(t)}{\partial \Delta\varepsilon_j^0(t)} + \frac{\partial U}{\partial \varepsilon_k(t)}\frac{\partial \varepsilon_k(t)}{\partial \Delta\varepsilon_j^0(t)}\right)\frac{\partial \Delta\varepsilon_j^0(t)}{\partial \mu_m} + \frac{\partial U}{\partial \mu_m}, \quad (3.2)$$

where the particular components can be expressed as follows:

$$\frac{\partial \Delta U}{\partial \sigma_p(t)} = \varepsilon_p(t)\mu_p A_p' l_p,$$

$$\frac{\partial U}{\partial \varepsilon_k(t)} = \sigma_k(t)\mu_k A_k' l_k, \qquad \frac{\partial U}{\partial \mu_m(t)} = \sum_t \sigma_m(t)\varepsilon_m(t) A_m' l_m,$$

$$\frac{\partial \sigma_p(t)}{\partial \Delta\varepsilon_j^0(t)} = E_p D_{pj}(t) + E_p \sum_{\tau \leqslant t} \sum_k \left(D_{pk}(t-\tau) - \delta_{pk} \right) \frac{\partial \Delta\beta_k^0(\tau)}{\partial \Delta\varepsilon_j^0(t)},$$

$$\frac{\partial \varepsilon_k(t)}{\partial \Delta\varepsilon_j^0(t)} = D_{pj}(t) + \sum_{\tau \leqslant t} \sum_k D_{pk}(t-\tau) \frac{\partial \Delta\beta_k^0(\tau)}{\partial \Delta\varepsilon_j^0(t)}.$$

The partial derivative $\frac{\partial \Delta\varepsilon_j^0(t)}{\partial \mu_m}$ can be determined from the following system of equations obtained thought differentiation of the formula (2.7) and by assuming (2.8), we get:

$$\frac{\partial \Delta\varepsilon_j^0(t)}{\partial \mu_m} = \varepsilon_j^L(t)\,\delta_{jm} + \sum_{\tau < t} \sum_n \left[(1 - \mu_j)\, D_{in}^D(t-\tau) - \delta_{in} \right] \frac{\partial \Delta\varepsilon_n^0(\tau)}{\partial \mu_m}.$$

In the second case we are looking for optimal distribution of yield stress limits in structural members and the design variables during optimisation process are σ_i^*. The corresponding gradient of the objective function, with respect to yield stress limits, takes the following form:

$$\frac{d\Delta U_i}{d\sigma_l^*} = \left(\frac{\partial \Delta U_i}{\partial \sigma_p(t)} \frac{\partial \sigma_p(t)}{\partial \Delta\beta_k^0(t)} + \frac{\partial \Delta U_i}{\partial \Delta\beta_k^0(t)} \right) \frac{\partial \Delta\beta_k^0(t)}{\partial \sigma_l^*}, \qquad (3.3)$$

where the new components of gradient can be expressed as follows:

$$\frac{\partial \sigma_p(t)}{\partial \Delta\beta_k^0(t)} = E_p D_{pk}(t) + E_p \sum_{\tau \leqslant t} \sum_j \left[D_{pj}(t-\tau) - \delta_{pj} \right] \frac{\partial \Delta\varepsilon_j^0(\tau)}{\partial \Delta\beta_k^0(t)}$$

The partial derivative $\frac{\partial \Delta\beta_k^0(t)}{\partial \sigma_l^*}$ can be determined from the following system of equations obtained through differentiation of the formula (2.12) and assumption of (2.8):

$$\frac{\partial \Delta\beta_k^0(t)}{\partial \sigma_l^*} = -\frac{(1-\gamma_k)}{E_k}\delta_{kl} + \sum_{\tau < t} \sum_m \left[(1-\gamma_k)\, H_{mk}(t-\tau) - \delta_{mk} \right] \frac{\partial \Delta\beta_m^0(\tau)}{\partial \sigma_l^*}.$$

Finally, the last case couples optimisation sub-problems: *remodelling* and *adaptation* of the structure. The design variables describe simultaneously material redistribution as well as yield stress limits: $\mu_i = \frac{A_i}{A_i'}$ and σ_i^*, respectively.

The coupled gradient formula takes the following form:

$$
\frac{dU}{d\sigma_l^*} = \left[\frac{\partial U}{\partial \sigma_p(t)} \left(\frac{\partial \sigma_p(t)}{\partial \Delta \varepsilon_j^0(t)} \frac{\partial \Delta \varepsilon_j^0(t)}{\partial \Delta \beta_k^0(t)} + \frac{\partial \sigma_p(t)}{\partial \Delta \beta_k^0(t)} \right) \right.
$$

$$
\left. + \frac{\partial U}{\partial \Delta \beta_k^0(t)} + \frac{\partial U}{\partial \mu_m} \frac{\partial \mu_m}{\partial \Delta \beta_k^0(t)} \right] \frac{\partial \Delta \beta_k^0(t)}{\partial \sigma_l^*}. \quad (3.4)
$$

4. Numerical Example

Simple truss structure (Fig. 2), has been chosen to prove functionality of the VDM based approach described above.

$E = 2.1e11\,\text{Pa}$
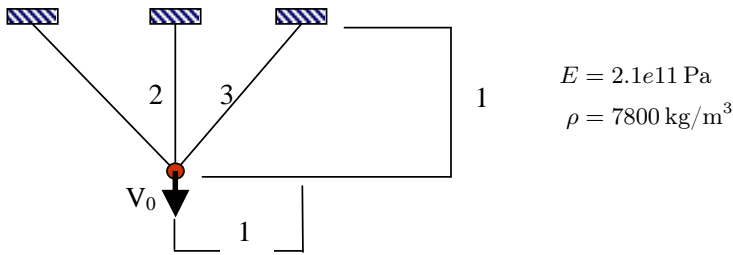
$\rho = 7800\,\text{kg/m}^3$

FIGURE 2. Testing example.

Let us assume first the yield stress limits for elements 1 and 3 are the same and equal to $\sigma_1^* = \sigma_2^* = 5 \cdot 10^8\,\text{Pa}$, while the limit for element 2 is equal to $\sigma_i^* = 4.5 \cdot 10^8\,\text{Pa}$. Verification of results obtained via the VDM-based approach versus analysis performed with use of ANSYS is demonstrated in Fig. 3. In the next example all elements have been assumed to have the same yield stress limits $\sigma_i^* = 5 \cdot 10^8\,\text{Pa}$, but different cross-sections, determined by the following parameters: $\mu_1 = \mu_3 = 0.7$ and $\mu_2 = 0.5$. The corresponding results are shown in Fig. 4.

5. Conclusions

- New approach to optimal design of dynamically loaded structures, based on the *Virtual Distortion Method* has been presented.
- This VDM-based approach allows for numerically efficient computation of analytically calculated gradients (with respect to mass distribution and yield stress limits).
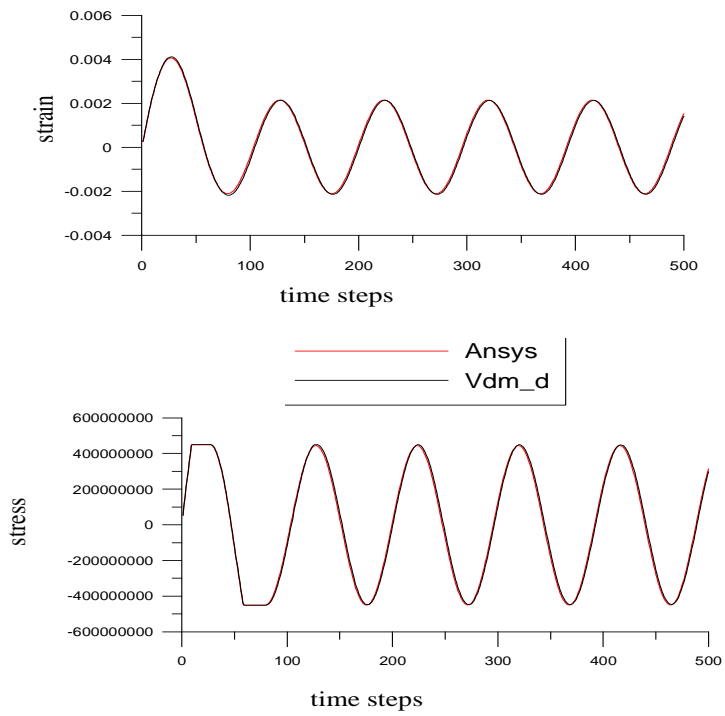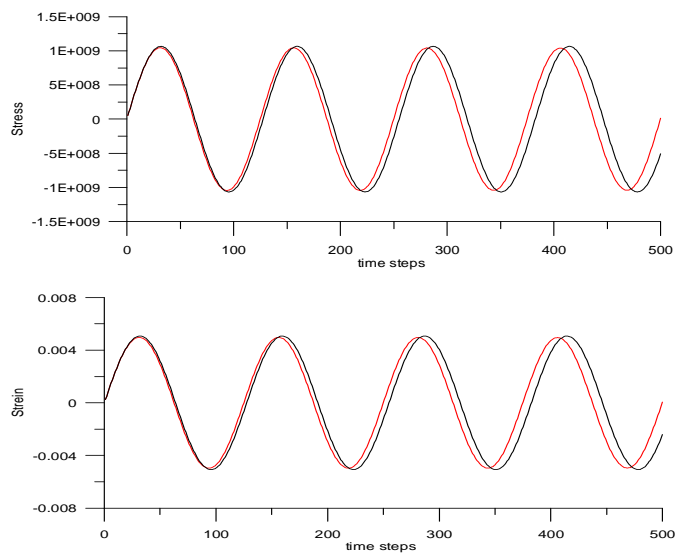
Figure 3. First testing example.

Figure 4. Second testing example.

- It is postulated that various, gradient-based optimal design problems can be effectively solved using algorithms based on the proposed approach (e.g. adaptation to maximal impact energy or the smoothest adaptation to determined impact).

- Further research is needed to develop corresponding, operational, numerical algorithms.

Acknowledgement

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